DIFFERENTIAL GRADED MANIFOLDS AND ASSOCIATED STACKS: AN OVERVIEW

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ABSTRACT. This is an overview of differential graded manifolds and their homotopy theory. The central topic is the construction of a functor from the category of dg manifolds to the homotopy category of simplicial presheaves on the site of smooth supermanifolds. This functor can be viewed as a generalization of the Lie functor from finite-dimensional Lie algebras to Lie groups.

1. A SUMMARY OF RESEARCH.

1.1. **A statement of purpose.** The ultimate goal of my ongoing research project is to develop a framework for studying the differential geometry of higher stacks; this includes, but is not limited to, "generalized differential geometry" in the sense of Hitchin [11]. Any differential geometry relies on infinitesimal methods; in the framework I would like to propose, these are based on the central notion of a *differential graded manifold* (dg manifold for short). In the world of stacks, dg manifolds are meant to play the role of infinitesimal objects (functions, tangent bundle, differential forms, etc.). Given a differentiable stack, one can associate to it a dg manifold in a natural way. For example, in case of the classifying stack of a Lie group, this procedure essentially reduces to obtaining the Lie algebra of the group; hence the term "higher Lie theory". As one might expect, the natural question is that of *integration*: does *every* dg manifold arise in this way? This question largely remains open. As I have been thinking about this problem for a while, it has become apparent that any solution is likely to make use not only of standard differential-geometric and analytic methods, but also, quite significantly, of abstract homotopical and higher categorical algebra. In particular, I am convinced (and hope to convince the reader as well) that the techniques of *derived/homotopical geometry*, as developed recently by Jacob Lurie, Bertrand Toën and their collaborators [15, 29] but adapted to the C^∞ setting, can be brought to bear on this problem. In what follows I outline the ideas and notions involved. The reader who merely wants to get a rough idea of what I have done and what I am trying to do need only read this Summary; the subsequent sections contain further details.

1.2. **Some (pre)history.** Given a smooth manifold M_0, its algebra of differential forms $\Omega(M_0)$ has for a long time been recognized as a fundamental tool for studying the geometry and topology of M_0. The algebra $\Omega(M_0)$ is naturally $\mathbb{N}$-graded and comes equipped with a canonical derivation d of degree 1 and square 0 – the de Rham differential. This algebra is universal among the differential graded commutative algebras $\mathcal{A}$ equipped with an algebra homomorphism $C^\infty(M_0) \to \mathcal{A}^0$.

The famous de Rham theorem asserts that the cohomology of the operator d is isomorphic to the (topological) cohomology of M_0 with real coefficients; furthermore, the work of Sullivan [28] has demonstrated that the real homotopy type of

M_0 can also be extracted from its algebra of differential forms, at least when the homotopy type of M_0 is nilpotent and finitely generated: one has to replace $\Omega(M_0)$ by an equivalent *minimal model*, which in many cases can be computed explicitly.

1.3. **Lie algebroids...** Although very successful in computing topological invariants of manifolds, Sullivan's methods do not directly apply to a wider class of differential graded algebras (dga's) of geometric origin which are very far from being connected, and so do not possess minimal models. A large subclass of these arises from *Lie algebroids*. Having roots in the work of Elie Cartan and Charles Ehresmann, these structures were originally defined by Jean Pradines [20] in terms of a Lie algebra bracket on sections of a vector bundle which is local of first order in each variable. Examples of Lie algebroids arise in a wide variety of situations in geometry: foliations, Lie algebra actions, Poisson structures on manifolds, to name a few [16, 17, 19]. Each Lie algebroid induces a (generally singular) foliation on the base manifold.

It was known for some time that a Lie algebroid can be equivalently described by a differential on the exterior algebra of the dual bundle, generalizing the de Rham complex of a manifold. Vaintrob [31] pointed out that it is this dual formulation that is best suitable for describing the category of Lie algebroids and various algebraic constructions therein; in order to retain the geometric intuition and keep the arrows pointing the right way, Vaintrob viewed the exterior algebras involved as algebras of functions on supermanifolds, and the differentials – as vector fields. This point of view eventually led to the notion of a *differential graded manifold* [33, 23].

1.4. **... and higher analogues.** *Courant algebroids* were introduced in [14] in an attempt to generalize the theory of Manin triples to Lie bialgebroids when it became clear that the category of Lie algebroids does not contain a solution. My own efforts to understand this result in Vaintrob's terms led me beyond exterior algebras to consider dga's with additional generators of degree 2 [22, 24]. Eventually I was able to characterize all Courant algebroids as dg manifolds of degree 2 with a compatible symplectic structure [23]. Such dg manifolds are examples of higher Lie algebroids.

Courant algebroids have since played a prominent role in the "generalized geometry" of the Hitchin school (motivated by string theory, see eg. [11]), as well as the geometry of the first Pontryagin class and vertex algebras [3] (See Examples 2.12, 2.13, 2.15).

In my most recent paper [21] I construct an explicit functor from Courant-Dorfman algebras – an algebraic generalization of Courant algebroids – to differential graded algebras, generalizing the correspondence obtained in [23], and study its properties.

Lastly, in our ongoing joint project with Paul Bressler, we have observed that truncating any vertex Poisson algebra yields a Courant-Dorfman algebra, and conversely, that any Courant-Dorfman algebra generates a vertex Poisson algebra in a canonical way, generalizing the Kac-Moody construction and showing that Courant-Dorfman algebras become Poisson structures in the chiral world.

Other examples of higher (and wider!) Lie algebroids are Mackenzie's double and triple Lie algebroids and bialgebroids ([17], see [32] for dg formulation).

1.5. **One differential vs. many brackets.** Traditionally, a Lie algebra is described by a bilinear bracket operation, whereas its higher, or homotopy-invariant

version – the L_∞-algebra – is similarly described by a sequence of multi-linear higher bracket operations obeying a system of quadratic equations generalizing the Jacobi identity. However, it has been observed that the same structure can be dually (and equivalently) described by a *single* differential on some graded commutative algebra; the brackets arise as its Taylor coefficients, as soon as coordinates are chosen. One point often overlooked is that the brackets are not invariant under non-linear automorphisms of the structure, only the total differential is; such differentials often arise naturally, and effortlessly [35].

There does exist a way in which brackets arise *naturally* from the differential, as *derived* brackets [12]; this generalizes Cartan calculus. However, when coordinates of higher degree are present, the brackets fail to satisfy the usual equations except in very special situations [34]. A careful analysis of the structure involved leads to higher categorical Lie algebras, as I show in [26]. For details, see Subsection 2.2.

The point of view that we adopt is that the notions "L_∞-algebroid" and "differential graded manifold" represent two ways of describing the same structure; we therefore feel justified in using the two terms as synonyms.

1.6. **Integration and homotopy.** The quotient of a manifold by a singular foliation – the coarse orbit space – is, in general, a nasty topological space which "remembers" very little or nothing of the original geometric situation (think of the Kronecker foliation on the torus as an example). Rather than simply identifying points in the same orbit, a more intelligent approach is to remember "all the reasons why" those points are equivalent (and ultimately also all equivalences between the equivalences, and so on); this line of reasoning eventually leads to higher groupoids and stacks. Now, the problem is that the dg manifold inducing the foliation only gives *infinitesimal* information: the directions towards equivalent points, as well as those virtual displacements keeping the given point fixed (the "internal degrees of freedom"). That is why it is important to find an answer to the following question: *does there exist some global object inducing the same foliation and having (infinitesimally) the same internal degrees of freedom at each point as the given dg manifold?*

For example, if the foliation is given by a Lie algebroid which comes from a Lie groupoid, one can construct a simplicial manifold, the *nerve* of the groupoid. One can then construct its geometric realization – the "Borel construction" – which has the "correct" homotopy type of the orbit space, the "homotopy quotient", as it takes into account the stabilizers of points, what I just called informally the "internal degrees of freedom". Alternatively, one can work with the cosimplicial dga of differential forms on the nerve, the Bott-Shulman double complex. Yet another approach is to consider the *differentiable stack* presented by the Lie groupoid which carries the same information as the homotopy quotient (see [18] for a nice review).

In view of this discussion, one may refine the above question by asking,

Question 1. *Can one find a simplicial manifold whose first-order approximation is the given dg manifold?*

The meaning of "first-order approximation" can now be made precise, thanks to the recent work of Pavol Ševera [35]. It was also Ševera who, inspired by Sullivan's work [28], first speculated that the object integrating a dg manifold might be its "fundamental Lie n-groupoid", understood in the appropriate sense [36]. But the breakthrough came with the work of Crainic and Fernandes [5] who constructed

the "fundamental groupoid" of a Lie algebroid and gave a precise answer to the question of when the resulting groupoid is smooth; this concluded several decades of research, beginning with the pioneering work of Mackenzie [16, 19]. The reason it may fail to be smooth is that the construction of the fundamental groupoid involves taking a coarse quotient (of "paths" by "homotopies"). If one instead remembers which paths are homotopic (while truncating the higher homotopies), one ends up with the holonomy stack of the homotopy foliation, which is smooth. Every choice of a complete transversal gives a presentation of the stack by a smooth finite-dimensional étale groupoid; the total structure is that of a Lie 2-groupoid [30]. The moral of this story is that, although a Lie algebroid may fail to be integrable by a Lie groupoid, it is always integrable by a Lie 2-groupoid (whose nerve is, in any case, a Kan simplicial manifold), but this 2-groupoid is only defined up to an equivalence more complicated than a mere isomorphism.

For higher Lie algebroids the question remains open, although good partial answers obtained by Hinich [10] (for nilpotent dg Lie algebras), Getzler [8] (for nilpotent L_∞-algebras) and Henriques [9] (for reduced finite-dimensional L_∞-algebras) seem to indicate the right way to proceed. However, the preceding discussion indicates that the issue of equivalence must be addressed (in particular, one must specify what "is" is in Question 1 above). So:

Question 2. *What is the appropriate notion of equivalence of dg manifolds, and of simplicial manifolds?*

It appears that the correct framework for dealing with these questions is that of simplicial presheaves on a site (of smooth manifolds) and Quillen model structures that exist there. We define a simplicial presheaf naturally associated to any dg manifold (equation (3.1)); one can then ask, for which l, if any, its homotopy l-type is "representable", i.e. isomorphic in the homotopy category to the presheaf associated to a finite-dimensional Lie l-groupoid. The question of equivalence can also be stated precisely in this framework. See Section 3.

1.7. **Sigma models.** By "abstract nonsense", the space of maps between any two dg manifolds is itself an infinite-dimensional dg manifold (this can be made precise in the language of presheaves). When the source and target dg manifolds are equipped with certain additional structure, this dg structure is realized by an action functional [1]. In [25] we show that dg manifolds coming from Courant algebroids have just such a structure and compute the extended BV action for the resulting 3-dimensional *Courant sigma model*.

On the other hand, the category of dg manifolds is also naturally enriched in simplicial presheaves (formula (4.1)). It appears that the dg structure is the infinitesimal version of this, although it is not yet clear how to make it precise.

1.8. **Plan.** In the following sections, I expand on the notions and ideas outlined above, trying to be precise without going into too much detail, and indicating my own contributions, where appropriate. In Section 2, I define dg manifolds, give several examples and sketch the basic properties of the geometries they describe. In Section 3, I formulate the integration problem for dg manifolds in the formalism of simplicial presheaves. This formulation appears to be new; while still far short of a solution, it at least has the advantage of asking the right questions. I also discuss how the solutions that already exist in special cases might be extended. The last

Section 4 touches upon a related issue about the infinitesimal structure of the space of maps between two dg manifolds and the global simplicial structure.

1.9. **Notation and conventions.** $\mathbb{R}$ denotes the field of real numbers. Unless otherwise indicated, all manifolds are real, finite-dimensional and smooth of class C^∞, but may have boundaries or corners, and may be supermanifolds; all functions, maps, tensor fields, etc. are smooth of class C^∞. For a (super)manifold M, $\mathcal{O}_M$ denotes its sheaf of smooth functions. If $E = \{E_i\}_{i\in\mathbb{Z}}$ is a graded vector bundle over M, $S(E)$ denotes the sheaf of graded-commutative $\mathcal{O}_M$-algebras freely generated by E; we use E also to denote the locally ringed space $(M, S(E^*))$, where E^* is the dual vector bundle (with opposite grading). For an integer k, $E[k]$ is the graded vector bundle with $E[k]_i = E_{i+k}$. We make no notational distinction between a vector bundle and its $\mathcal{O}$-module of sections, except for the occasional use of Γ to denote global sections.

2. Differential graded manifolds and their associated geometries.

2.1. Definition and some examples.

Definition 2.1. A (nonnegatively) *graded manifold* is a locally ringed (super)space (more precisely, a space locally ringed by graded commutative $\mathbb{R}$-algebras), $M = (M_0, \mathcal{O}_M)$, which is locally isomorphic to $(U, \mathcal{O}_U \otimes S(V))$, where $U \subset \mathbb{R}^{m|r}$ is an open (super)domain and $V = \{V_i\}_{0<i\leq n}$ is a finite-dimensional positively graded (super)vector space.

A *differential graded manifold* (dg manifold) is a graded manifold equipped with a global section d of $\mathrm{Der}^1(\mathcal{O}_M)$ satisfying $d^2 = [d,d]/2 = 0$.

Morphisms of dg manifolds are defined to be the morphisms of locally ringed spaces respecting the differentials. The resulting category is denoted by **dgMan**.

Remark 2.2. In particular, the base M_0 is a smooth (super)manifold, and any dg map induces a smooth map of the bases. It is convenient to think of V above as the dual of some *negatively* graded vector space; when M_0 is connected, the number n appearing in the above definition – the highest degree of a local generator – is an invariant, referred to as the *degree* of M. We think of sections of $\mathcal{O}_M$ as functions on M, sections of $\mathrm{Der}(\mathcal{O}_M)$ as vector fields on M, and so on. See [33] for an introduction to the theory of graded manifolds.

Example 2.3. Given a manifold M_0, the graded manifold $T[1]M_0 = (M_0, \Omega(M_0))$ is referred to as the *odd tangent bundle* of M_0. The dg structure is provided by the de Rham differential. The association $M_0 \mapsto T[1]M_0$ extends to a fully faithful functor. The universal property of differential forms can be expressed by saying that this functor is right adjoint to the "0th truncation" functor assigning to each dg manifold M its base M_0. The unit of adjunction gives a canonical dg map $M \to T[1]M_0$, natural in M, called the *anchor map.* This example is fundamental.

Example 2.4. The 0th truncation functor also has a left adjoint, associating to a manifold M_0 a dg manifold $(M_0, C^\infty_{M_0})$, with zero grading and differential.

Example 2.5. Given a Lie algebra $\mathfrak{g}$, define $\mathfrak{g}[1] = (*, S(\mathfrak{g}[1]^*) = \wedge\mathfrak{g}^*)$, with dg structure given by the Chevalley-Eilenberg differential.

Example 2.6. Given an action of a Lie algebra $\mathfrak{g}$ on a manifold M_0, define $A = M_0 \times \mathfrak{g}$ and let $A[1] = (M_0, \mathcal{O}_{M_0} \otimes S(\mathfrak{g}[1]^*))$, with the dg structure given by the

Chevalley-Eilenberg differential with coefficients in the $\mathfrak{g}$-module $C^\infty(M_0)$ (often referred to as "BRST operator" in the physics literature).

Example 2.7. Let M_0 be a Poisson manifold, with Poisson tensor $\pi \in \Gamma(\wedge^2 TM_0)$. In view of the Jacobi identity $[\pi, \pi] = 0$, the *odd cotangent bundle* $T^*[1]M_0 = (M_0, \wedge TM_0)$ becomes a dg manifold with the Lichnerowicz differential $d = [\pi, \cdot]$, where $[\cdot, \cdot]$ denotes the Schouten bracket of multivector fields. This bracket comes from the canonical symplectic structure on $T^*[1]M_0$ with respect to which d is a Hamiltonian vector field.

Example 2.8. Let $A \to M_0$ be a vector bundle, and let $A[1] = (M_0, S(A[1]^*) = \wedge A^*)$. Any $\mathbb{N}$-graded manifold of degree 1 is of this form; any dg structure on $A[1]$ is equivalent to a *Lie algebroid* structure on the bundle A [31]. All the preceding examples are special cases of this one.

Before proceeding with examples of dg manifolds of higher degree, let us pause to make the following observations. Each dg manifold (M, d) of degree n has a canonically associated complex of vector bundles

$$\mathbb{T}M = \{A_{-n} \xrightarrow{\delta_{-n}} \cdots \xrightarrow{\delta_{-2}} A_{-1} \xrightarrow{\delta_{-1}} TM_0\} \tag{2.1}$$

called the *tangent complex* of M. The differential δ is (dual to) the linear part of d. The last map δ_{-1} is the anchor map; in fact, this complex generalizes the anchor map $A_0 = A \to TM_0$ of a Lie algebroid. The image of δ_{-1} is an involutive module of vector fields, i.e. a singular foliation on M_0. We call M *reduced* if M_0 is a point; *connected* or *transitive* if δ_{-1} is surjective; *regular* if all δ_i's have constant ranks; *exact* if its tangent complex is acyclic. Expanding d in a Taylor series at a point $m \in M_0$ gives rise to an L_∞-algebra structure on $\mathbb{T}_m M[-1]$ (i.e. a "Lie algebra up to higher homotopies" in the sense of Stasheff [13]).

Remark 2.9. Any $\mathbb{N}$-graded manifold comes equipped with a canonical projection $M \to M_0$ and the zero section $M_0 \to M$. If $A := \{A_i\}_{-n \leq i < 0}$ in (2.1) above, A is the normal bundle to the zero section. The total space of A is *not* the same as M (except when $n = 1$), although in the C^∞ category they are always (non-canonically) isomorphic. In spite of that, I choose not to define graded manifolds in this way, as such an isomorphism is almost never part of the data; besides, in the algebraic or analytic setting it does not generally exist at all, while the definition I have given still applies.

Remark 2.10. A reduced dg manifold is the same thing as a finite-dimensional L_∞-algebra concentrated in degrees $(-\infty, 0]$ (the degree shift is customary, so that ordinary Lie algebras end up in degree 0). Actually, it is concentrated in degrees $(-n, 0]$ where n is the degree of the graded manifold, so the L_∞-algebra is actually a Lie n-algebra. Another name for a dg manifold of degree n that is not reduced is "Lie n-algebroid".

Example 2.11. Let $\mathfrak{g}$ be a simple Lie algebra of compact type, $\langle \cdot, \cdot \rangle$ its Killing form, and let $\Theta = \langle [\cdot, \cdot], \cdot \rangle$. Define a dg structure on $\mathfrak{g}[1] \oplus \mathbb{R}[2]$ as follows:

$$d = d_{\mathfrak{g}} + \Theta \frac{\partial}{\partial t}$$

where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$ and t is the coordinate on $\mathbb{R}[2]$. The corresponding Lie 2-algebra structure on $\mathbb{R}[1] \oplus \mathfrak{g}$ is denoted by $\mathfrak{str}$ and called the *string Lie 2-algebra* for its rôle in string theory.

Example 2.12. Given a vector bundle $E \to M_0$, a *Courant algebroid* structure on E consists of a non-degenerate symmetric $\mathcal{O}_{M_0}$-bilinear form $\langle\cdot,\cdot\rangle$ on E and an $\mathbb{R}$-bilinear bracket $[\cdot,\cdot]$ on sections of E, such that for every section e of E, the operator $[e,\cdot]$ is a derivation of E, $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ itself. The latter means that $[\cdot,\cdot]$ is a Leibniz bracket; one requires, in addition that its symmetric part be equal to the differential of $\langle\cdot,\cdot\rangle$ (see [23] for details).

Define M to be the pullback of the diagram $E[1] \to (E \oplus E^*)[1] \leftarrow T^*[2]E[1]$, where the left arrow is induced by the canonical isometric embedding $E \to E \oplus E^*$, and the right one is the canonical projection. M is an affine bundle over $E[1]$ under the action of $T^*[2]M_0$, hence a graded manifold of degree 2 (unless M_0 is a point); it also inherits a symplectic structure from $T^*[2]E[1]$. It was proved in [23] that Courant algebroid structures on E correspond exactly to global sections Θ of $\mathcal{O}^3_M$ satisfying $\{\Theta,\Theta\} = 0$. M then becomes a dg manifold with $d = \{\Theta,\cdot\}$, i.e. a Lie 2-algebroid (compare with Example 2.7). The tangent complex (2.1) of M takes the form

$$T^*M_0 \xrightarrow{a^*} E \xrightarrow{a} TM_0 \tag{2.2}$$

where a is the anchor.

It is possible to describe the algebra of functions on M explicitly in terms of operations on the bundle E (see [21] where we work in a general algebraic setting); in particular, Θ can be defined by the formula $\Theta(x,y,z) = \langle [x,y],z\rangle$ for $x,y,z \in \Gamma(E)$. When M_0 is a point, a Courant algebroid is just a Lie algebra with an invariant inner product, and Θ is as in Example 2.11.

Example 2.13. Let P be a principal $\mathbb{R}[2]$-bundle over $T[1]M_0$ in the category of dg manifolds, where $\mathbb{R}[2]$ has the zero differential and $T[1]M_0$ the standard one of Example 2.3. Such principal bundles are classified by $H^3(M_0,\mathbb{R})$, as can be easily seen by choosing a global section [36]. The induced $\mathbb{R}[2]$-action on $T^*[2]P$ is strongly Hamiltonian with momentum map μ taking values in $\mathbb{R}$ (without degree shift!). Let $M = \mu^{-1}(1)/\mathbb{R}[2]$ be the Marsden-Weinstein quotient. The induced differential on M is Hamiltonian, and the resulting Courant algebroid (defined on $E = \mathcal{O}^1_M = \mathrm{Vect}^{-1}(P)$) is *exact*, in the sense that its tangent complex (2.2) is a short exact sequence. It can be shown that all exact Courant algebroids arise in this way.

Example 2.14. Let A be a Lie algebroid over M_0. Define its *Weil algebroid* $W(A)$ to be the graded manifold $T[1]A[1]$ with the differential equal to the sum of the de Rham differential on forms on $A[1]$ and the canonical lift (the Lie derivative) of the differential on $A[1]$ inducing the Lie algebroid structure (Example 2.8). $W(A)$ is a Lie 2-algebroid, with tangent complex

$$A \longrightarrow A \oplus TM_0 \longrightarrow TM_0 \tag{2.3}$$

where the first map is the inclusion, the second – the projection. The algebra of functions on $W(A)$ can be described explicitly [2] (the approach there has the same flavor as in [21]); when M_0 is a point, it is the well-known Weil algebra associated to a Lie algebra.

2.2. **Differential calculi and generalized geometries.** As discovered by Cartan and further elaborated by Frölicher and Nijenhuis [7], differential calculus on a manifold can be described completely in terms of its algebra of differential forms.

The space $\mathrm{Vect}(T[1]M_0) = \mathrm{Der}\Omega(M_0)$ is a dg Lie algebra under the graded commutator bracket $[\cdot,\cdot]$ and the differential $D = [d,\cdot]$. In particular, $\mathrm{Vect}(M_0)$ can be identified with $\mathrm{Vect}^{-1}T[1]M_0$ by associating to a vector field v on M_0 the contraction operator ι_v. Obviously, $[\iota_v, \iota_w] = 0$ (because there is nothing in degree -2), but one has the Cartan formula:

$$[i_v, [d, \iota_w]] = \iota_{\{v,w\}} \tag{2.4}$$

where $\{\cdot,\cdot\}$ denotes the commutator of vector fields on M_0. An expression of the form $[\cdot, D\cdot]$ is called a *derived bracket* [12, 33], and $D\iota_w = [d, \iota_w] = L_w$ is the Lie derivative operator. The space of infinitesimal automorphisms of the dg manifold $T[1]M_0$ consists of those elements in $\mathrm{Vect}^0(T[1]M_0)$ which commute with d, i.e. 0-cocycles with respect to D. They form a Lie subalgebra under $[\cdot,\cdot]$, denoted by $\mathrm{Vect}^0_0(T[1]M_0)$. However, it was shown in [7] that the differential D is contractible; in particular, all infinitesimal automorphisms of $T[1]M_0$ as a dg manifold are the Lie derivatives by vector fields on M_0. In fact, the automorphism group of $T[1]M_0$ coincides with the diffeomorphism group of M_0 – it is an immediate consequence of the more general fact that $T[1]$ is a fully faithful functor.

These considerations generalize, to a large extent, to other dg manifolds, each thus defining its own differential calculus. Given a dg manifold M, we can consider the dg Lie algebra $\mathrm{Vect}(M)$ controlling the deformation theory of (M, d). For M of degree n, $\mathrm{Vect}(M)$ is concentrated in degrees $[-n, +\infty)$. When $M = A[1]$ is a Lie algebroid, the calculus is very similar to the classical one: again one has $\mathrm{Vect}^{-1}(A[1]) \simeq \Gamma(A)$ and the Cartan formula (2.4) holds, with $\{\cdot,\cdot\}$ now denoting the bracket of sections in the Lie algebroid [16]. The only difference is that the cohomology of D may be nontrivial, and consequently not all infinitesimal automorphisms of $(A[1], d)$ are the Lie derivatives by sections of A. Instead, one has a *Lie algebra crossed module*:

$$\Gamma(A) \xrightarrow{D} \mathrm{Vect}^0_0(A[1])$$

i.e. D is a Lie algebra homomorphism, $\mathrm{Vect}^0_0(A[1])$ acts on $\Gamma(A)$, and the bracket on $\Gamma(A)$ is derived.

The novel feature for $n > 1$ is that, while $\mathrm{Vect}^{<0}(M)$ is closed under $[\cdot,\cdot]$, the restriction of $[\cdot,\cdot]$ does not vanish, and consequently, the derived bracket only defines a Leibniz, rather than Lie, algebra. The structure of the resulting algebra was analyzed in detail in [26] for $n = 2$. For Courant algebroids, the answer was already contained in [27]; when the base of the algebroid is a point, it reduces to Example 2.11.

Example 2.15. Consider $P = P_{[\phi]}$ as in Example 2.13, associated to a class $[\phi] \in H^3(M_0, \mathbb{R})$. One has $\mathrm{Vect}^{-2}(P) \simeq \mathcal{O}_{M_0}$ via $f \mapsto f\frac{\partial}{\partial t}$. Let $E = \mathrm{Vect}^{-1}(P)$; it becomes a Courant algebroid with $\langle\cdot,\cdot\rangle = [\cdot,\cdot]$ and the Dorfman bracket the derived bracket $[\cdot, [d,\cdot]]$. This Courant algebroid is exact, its Ševera class is equal to $[\phi]$. The automorphism group of P is an extension of the diffeomorphism group of M_0 by the abelian group of closed 2-forms on M_0 (these are precisely the dg maps from $T[1]M_0$ to $\mathbb{R}[2]$). Generalized geometry in the sense of Hitchin – "differential geometry in the presence of a gerbe" – is is based on the differential calculus associated to $P_{[\phi]}$ as a dg manifold.

2.3. **DG structure as an action of $\mathbf{Hom}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$.** Every dg manifold becomes a supermanifold, if one forgets the differential and only remembers the grading modulo 2, which distinguishes commuting functions from anticommuting. What extra structure on a supermanifold makes it a dg manifold? Recall that the *odd affine line* is the supermanifold $\mathbb{A}^{0|1}$, the principal homogeneous space for the group object $\mathbb{R}^{0|1} = (*, \wedge(\mathbb{R}))$, where $\wedge(\mathbb{R}) = \mathbb{R}[\tau]/(\tau^2)$ is the exterior algebra on one generator τ, viewed as a $\mathbb{Z}_2$-graded algebra with τ odd. As the definition makes intuitively clear, the odd line can be used as a "probe" to extract first-order information from geometric objects. What distinguishes it crucially from the "double point" – which has the same algebra of functions and is used for the same purpose in algebraic geometry – is the $\mathbb{Z}_2$-grading. Namely, $d/d\tau$ is a well-defined *odd* derivation, so translation by τ makes sense; in fact, the odd line behaves in all respects like its even counterpart, the real affine line, while still remaining "infinitesimal".

Given supermanifolds X and Y, define $\mathbf{Hom}(X, Y)$ to be the presheaf on the category of supermanifolds sending U to the set $\mathrm{Hom}(U \times X, Y)$. When this presheaf is representable, denote the representing supermanifold also by $\mathbf{Hom}(X, Y)$. Because of infinite dimensionality, this is never the case unless X has zero even dimension; for the odd line, the representing object turns out to be rather well-known: $\mathbf{Hom}(\mathbb{A}^{0|1}, Y)$, the space of odd curves in Y, is nothing but $T[1]Y$! It is acted upon naturally on the right by the monoid object $\mathbf{Hom}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$. The structure of this monoid is easy to describe: it is isomorphic to $\mathbb{M}_m(\mathbb{R}) \rtimes \mathbb{R}^{0|1}$, where $\mathbb{M}_m$ denotes the multiplicative monoid. This explains the canonical structure we have on differential forms: the dilation action of $\mathbb{M}_m$ gives rise to the $\mathbb{N}$-grading, while the de Rham differential is the generator of the translations by $\mathbb{R}^{0|1}$. It can be shown that, in general, a dg structure on a supermanifold is the same thing as a right action of $\mathbf{Hom}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$. Among other things, this explains the universal property of differential forms.

The above facts have been known to the cognoscenti for some time, but in [35] the idea of probing with the odd line is taken seriously, and it is shown that dg manifolds arise as first-order approximations of just about everything in the world. The basic idea is very simple. Suppose $j : Man \to \mathcal{C}$ is a fully faithful product-preserving functor into some category $\mathcal{C}$ and C is an object of $\mathcal{C}$; if $\mathrm{Hom}(j(\mathbb{A}^{0|1}), C)$ is a manifold, it has a natural dg structure (to make this precise one must extend j to the category of *families* of manifolds). For instance, one can choose $\mathcal{C}$ to be the category of simplicial manifolds, with $j(X) = \Delta^X = ([n] \mapsto X^{n+1})$. The main result of [35] is that if C is a Kan simplicial manifold, the space of simplicial maps from $\Delta^{\mathbb{A}^{0|1}}$ into C is a manifold, hence a dg manifold. For example, applying this construction to the nerve of a Lie group(oid) yields the dg manifold associated to its Lie algebra(oid). The natural question is, then: *does every dg manifold arise in this way?*

3. Integrating dg manifolds.

3.1. **The case of Lie algebras.** In an attempt to answer the preceding question, one can begin with the case of a Lie algebra $\mathfrak{g}$ and take a cue from Sullivan [28] who considered the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on Euclidean simplices satisfying the Maurer-Cartan equation:

$$\mathrm{MC}_n(\mathfrak{g}) = \{\alpha \in \Omega^1(\Delta^n, \mathfrak{g}) | d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}$$

This is in fact a Kan complex; if $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, the fundamental group of $\mathrm{MC}_\bullet(\mathfrak{g})$ is $\tilde{G}_{disc}$, the universal cover of G with discrete topology, while its higher homotopy groups are those of G. Thus, we obtain the simply connected Lie group integrating $\mathfrak{g}$ *as a discrete group*; to recover the smooth structure, one observes that, as long as forms of class C^r are used (with arbitrary but fixed r), $\mathrm{MC}_\bullet(\mathfrak{g})$ is in fact a Kan simplicial Banach manifold, and its fundamental group $\pi_1(\mathrm{MC}_\bullet(\mathfrak{g}))$ is the quotient of the Banach manifold $\mathrm{MC}_1(\mathfrak{g})$ by a simple foliation of codimension equal to the dimension of g, and thus inherits a smooth structure making it a Lie group. One can then view the 1-truncation of $\mathrm{MC}_\bullet(\mathfrak{g})$ – the nerve of $\tilde{G}$ – as the simplicial manifold integrating the dg manifold $\mathfrak{g}[1]$.

3.2. **The general case.** Again following Sullivan, we define, for a given dg manifold M, a simplicial set $X_\bullet^M$:

$$[n] \mapsto X_n^M = \mathrm{Hom}_{\mathbf{dgMan}}(T[1]\Delta^n, M).$$

For $M = \mathfrak{g}[1]$, this is the same as $\mathrm{MC}_\bullet(\mathfrak{g})$, so it is natural to look for an object integrating M inside it. Unfortunately, while it can be shown (by restricting to an orbit) that $X_\bullet^M$ is a Kan simplicial set, it is unknown whether it is naturally a Banach manifold in general.

In the special case $M = A[1]$ with A a Lie algebroid, Crainic and Fernandes [5] construct its fundamental groupoid "by hand", using path concatenation with a cutoff function; they prove that when this groupoid is smooth, it is the Lie groupoid integrating A. Its nerve – the 1-truncation of $X_\bullet^{A[1]}$ – is then the simplicial manifold integrating $A[1]$. It is further shown in [30] that for *any* Lie algebroid there is a (non-canonical) finite-dimensional Lie 2-groupoid equivalent to the 2-truncation of $X_\bullet^{A[1]}$; any such 2-groupoid (or rather, the stack it represents) can be viewed as the integrating object for $A[1]$

In another direction, Henriques [9] shows that, when M is a *reduced* dg manifold – i.e., of the form $\mathfrak{g}[1]$ for a finite-dimensional Lie n-algebra $\mathfrak{g}$ – $X_\bullet^{\mathfrak{g}[1]}$ is a Kan simplicial Banach manifold. The proof uses the Postnikov tower of $\mathfrak{g}$. It is further proved that, while the nth truncation of $X_\bullet^{\mathfrak{g}[1]}$ may fail to be smooth, the (n+1)st one always is. This still leaves open the question of whether there exists a *finite-dimensional* simplicial manifold equivalent to it. This is not known even for the string Lie 2-algebra (Example 2.11).

It seems likely that Henriques's methods can be extended to *regular* dg manifolds, as those do possess a Postnikov tower. But beyond that, a new approach is needed.

3.3. **Simplicial presheaves.** In order to avoid dealing with infinite-dimensional spaces and provide a proper setting in which to discuss equivalence of integrating objects, I propose to "parametrize" the above construction as follows. Given a dg manifold M, define a presheaf $\Pi_\infty(M)$ of simplicial sets on the site of smooth finite-dimensional (super)manifolds by

$$\Pi_\infty(M)_n(B) = \mathrm{Hom}_{\mathbf{dgMan}}(B \times T[1]\Delta^n, M) \tag{3.1}$$

where the B on the right is viewed as a dg manifold with the zero dg structure. In particular, $\Pi_\infty(M)_n(*) = X_n^M$; heuristically, $\Pi_\infty(M)_n(B)$ would be the set of smooth maps from B into X_n^M, if the latter were a finite-dimensional manifold. The presheaf $\Pi_\infty(M)$ should be viewed as a replacement for this nonexistent manifold

structure. Here we view the category of smooth supermanifolds as a Grothendieck site with respect to the usual "open covers" pretopology.

For any simplicial presheaf on a site one has the associated sheaves of homotopy groups (sets in case of π_0). The homotopy sheaves of $\Pi_\infty(M)$ are important invariants of the dg manifold M. In special cases these sheaves are representable (by Lie groups), as e.g. in the case of $M = \mathfrak{g}[1]$ for a Lie algebra $\mathfrak{g}$. It follows from the work [36] that these groups are discrete in all dimensions higher than the degree of M.

There exists a Quillen model structure on the category of simplicial presheaves on a site for which the weak equivalences are those maps which induce isomorphisms of all homotopy sheaves (the *local equivalences*), while the fibrant objects are those presheaves which are (1) objectwise fibrant (i.e. valued in Kan complexes) and (2) satisfy descent for hypercovers [6]. The fibrant presheaves are "nice" from the homotopy-theoretic viewpoint, as their homotopy groups can be computed combinatorially, and also satisfy a certain local-to-global property, i.e. are, in some sense, sheaves. One defines an *infinity-stack* (or simply a *stack*) to be such a fibrant simplicial presheaf [29].

The presheaf $\Pi_\infty(M)$ should be viewed as an object in the associated homotopy category of stacks. As we have already remarked, it is very likely to be objectwise fibrant, although it has only been proven in some special cases. Assuming that, one can apply Duskin's truncation [8] and obtain, for each positive integer l, the *fundamental l-groupoid* of M, denoted by $\Pi_l(M)$. It is a stack whose homotopy sheaves in dimensions up to l are isomorphic to those of $\Pi_\infty(M)$, and all the higher ones vanish.

On the other hand, to any simplicial manifold $X_\bullet$ one can associate a simplicial presheaf

$$\mathfrak{X}_n(B) = \mathrm{Hom}_{\mathbf{Man}}(B, X_n)$$

and then apply fibrant replacement to get a stack. Those stacks which arise in this way from Lie l-groupoids (in the sense of [9]) are precisely what one would call *differentiable l-stacks*.

Definition 3.1. A dg manifold M is called *l-integrable* if there exists an $l \geq n = \deg(M)$ such that $\Pi_l(M)$ is isomorphic in the homotopy category of stacks to $\mathfrak{X}_\bullet$ for some Lie l-groupoid $X_\bullet$. M is called *integrable* if it is n-integrable.

For instance, for Lie algebroids this notion of integrability coincides with the usual one; all Lie algebroids are 2-integrable. The central problem in higher Lie theory is to determine the obstructions to integrability for all dg manifolds.

4. Mapping spaces and sigma-models.

DG manifolds form a simplicial presheaf-enriched category by setting

$$\mathrm{Hom}(M,N)_n(B) = \mathrm{Hom}(B \times M \times T[1]\Delta^n, N) \tag{4.1}$$

(compare with (3.1)) In particular, this gives a notion of homotopy between two dg maps; in case of Lie algebroids, this notion was used in [5] in a crucial way. In that paper, a central role was played by the corresponding *infinitesimal* notion: those variations of a given map (of Lie algebroids) which point in the direction of homotopic ones.

It turns out that this information (and much more) can be nicely encoded in a dg structure that exists on the space of maps. The infinite-dimensional and generally

badly singular $\mathrm{Hom}_{\mathbf{dgMan}}(M, N)$ can be interpreted as the fixed-point set of this dg structure. Recall that a non-negatively graded dg structure is the same thing as a right action of $\mathbf{Hom}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$ on a supermanifold. When M and N are viewed as merely (super)manifolds, the presheaf $\mathbf{Hom}(M, N)$ may not be representable, but it still makes sense to talk about an action of a monoid or group object on it. In particular, while the actions of the monoid $\mathbf{Hom}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$ on M and N do not induce one on $\mathbf{Hom}(M, N)$, the actions of the group $\mathbf{Iso}(\mathbb{A}^{0|1}, \mathbb{A}^{0|1})$ do. What this means is that $\mathbf{Hom}(M, N)$ has a natural dg structure, except it is $\mathbb{Z}$- rather than $\mathbb{N}$-graded. This dg structure contains information not only about dg maps (which form its fixed-point set), but also the infinitesimal gauge transformations (homotopies), relations among them, and so on. When the target N has a compatible symplectic structure, while the source M has a compatible measure (e.g. $M = T[1]M_0$), the dg structure on $\mathbf{Hom}(M, N)$ is Hamiltonian, represented by an action functional, as first observed in [1]. This leads to interesting topological field theories [4, 25].

Investigating the relation between the dg structure on $\mathbf{Hom}(M, N)$ and the simplicial structure on $\mathrm{Hom}(M, N)_\bullet$ is likely to lead to interesting insights.

References

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the Master equation and topological quantum field theory*, Int. J. Modern Phys. A **12** (1997), no. 7, 1405–1429.

[2] C. Arias Abad and M. Crainic, *The Weil algebra and the van Est isomorphism*, arXiv:0901.0322.

[3] P. Bressler, *The first Pontryagin class*, Compos. Math. **143** (2007), no. 5, 1127–1163, arXiv:math/0509563.

[4] A. Cattaneo and G. Felder, *On the AKSZ formulation of the Poisson sigma model*, Lett. Math. Phys. **56** (2001), 163–179.

[5] M. Crainic and R.L. Fernandes, *Integrability of Lie brackets*, Ann. Math. **157(2)** (2003), 575–620.

[6] D. Dugger, S. Hollander, and D. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Phil. Soc. **136** (2004), 9–51, arXiv:math/0205027v2.

[7] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms, part I*, Indag. Math **18** (1956), 338–359.

[8] E. Getzler, *Lie theory for nilpotent L_∞-algebras*, math.AT/0404003, to appear in Ann. Math.

[9] A. Henriques, *Integrating L_∞-algebras*, Compos. Math. **144** (2008), no. 4, 1017–1045.

[10] V. Hinich, *Descent for Deligne groupoids*, Internat. Math. Res. Notices (1997), 223–239, arXiv:alg-geom/9606010.

[11] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford Ser. **54** (2003), 281–308.

[12] Y. Kosmann-Schwarzbach, *From Poisson algebras to Gerstenhaber algebras*, Ann. Inst. Fourier, Grenoble **46** (1996), no. 5, 1243–1274.

[13] T. Lada and J. Stasheff, *Introduction to SH Lie algebras for physicists*, International Journal of Theoretical Physics **32(7)** (1993), 1087–1103.

[14] Zhang-Ju Liu, Alan Weinstein, and Ping Xu, *Manin triples for Lie bialgebroids*, J. Diff. Geom. **45** (1997), 547–574.

[15] J. Lurie, *Derived algebraic geometry I-IV*, available on arXiv.org.

[16] K. Mackenzie, *Lie Groupoids and Lie Algebroids in differential geometry*, LMS lecture notes series, vol. 124, Cambridge Univ. Press, 1987.

[17] ______, *General theory of Lie groupoids and Lie algebroids*, LMS lecture notes series, vol. 213, Cambridge Univ. Press, 2005.

[18] D. Metzler, *Topological and smooth stacks*, arXiv:math/0306176, 2003.

[19] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge studies in advanced mathematics, vol. 91, Cambridge Univ. Press, 2003.

[20] J. Pradines, *Géométrie differentielle au-dessus d'un groupoïde*, C. R. Acad. Sci. Paris, série A **266** (1968), 1194–1196.
[21] D. Roytenberg, *Courant-Dorfman algebras and their cohomology*, Lett. Math. Phys., 2009, DOI 10.1007/s11005-009-0342-3, arXiv:0902.4862.
[22] ______, *Courant algebroids, derived brackets and even symplectic supermanifolds*, Ph.D. thesis, UC Berkeley, 1999, math.DG/9910078.
[23] ______, *On the structure of graded symplectic supermanifolds and Courant algebroids*, Quantization, Poisson Brackets and Beyond (Theodore Voronov, ed.), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, math.SG/0203110.
[24] ______, *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Lett. Math. Phys. **61(2)** (2002), 123–137, math.QA/0112152.
[25] ______, *AKSZ-BV formalism and Courant algebroid-induced topological field theories*, Lett. Math. Phys. **79(2)** (2007), 143–159, see also hep-th/0608150.
[26] ______, *On weak Lie 2-algebras*, XXVI Workshop on Geometrical Methods in Physics (Piotr Kielanowski et. al., ed.), AIP Conference Proceedings, vol. 956, American Institute of Physics, Melville, NY, 2007, arXiv:0712.3461; also appeared in the MPI and ESI preprint series.
[27] D. Roytenberg and A. Weinstein, *Courant algebroids and strongly homotopy Lie algebras*, Lett. Math. Phys. **46** (1998), 81–93, math.QA/9802118.
[28] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. IHÉS **47** (1977), 269–331.
[29] B. Toën, *Higher and derived stacks: a global overview*, arXiv:math/0604504.
[30] Hsian-Hua Tseng and Chenchang Zhu, *Integrating Lie algebroids via stacks*, Compos. Math. **142(1)** (2006), 251–270.
[31] A. Vaintrob, *Lie algebroids and homological vector fields*, Uspekhi Mat. Nauk **52(2)** (1997), 428–429.
[32] T. Voronov, *Mackenzie theory and Q-manifolds*, arXiv:math/060811.
[33] ______, *Graded manifolds and Drinfeld doubles for Lie bialgebroids*, Quantization, Poisson Brackets and Beyond (Theodore Voronov, ed.), Contemp. Math., vol. 315, Amer. Math. Soc., Providence, RI, 2002, Preprint math.DG/0105237.
[34] ______, *Higher derived brackets and homotopy algebras*, J. Pure Appl. Algebra **202** (2005), no. 1–3, 133–153.
[35] P. Ševera, *L_∞-algebras as 1-jets of simplicial manifolds (and a bit beyond)*, preprint math.DG/0612349.
[36] ______, *Some title containing the words "homotopy" and "symplectic", e.g. this one*, Travaux Mathématiques, vol. XVI, Univ. Luxemb., Luxembourg, 2005, also available as math.SG/0105080, pp. 121–137.